

CANONICAL BASIS FOR QUANTUM  $\mathfrak{osp}(1|2)$ 

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ABSTRACT. We introduce a modified quantum enveloping algebra as well as a (modified) covering quantum algebra for the ortho-symplectic Lie superalgebra  $\mathfrak{osp}(1|2)$ . Then we formulate and compute the corresponding canonical bases, and relate them to the counterpart for  $\mathfrak{sl}(2)$ . This provides a first example of canonical basis for quantum superalgebras.

## 1. INTRODUCTION

The canonical basis of Lusztig [11] and Kashiwara [9] has served as an important motivation of the categorification of quantum enveloping algebras. In a recent paper [5] of David Hill and the second author, a class of (halves of) quantum Kac-Moody *superalgebras* has been categorified, and in addition, it was suggested for the first time to use a novel bar-involution to construct canonical basis of quantum Kac-Moody superalgebras and their integrable modules. We refer the reader to *loc. cit.* for extensive references in the fast-growing area of categorification.

The aim of this paper is to formulate and compute the canonical bases for a modified quantum enveloping superalgebra  $\dot{U}$  as well as for a (modified) covering quantum superalgebra  $\dot{U}^\pi$  associated to the ortho-symplectic Lie superalgebra  $\mathfrak{osp}(1|2)$ . Since canonical basis has never been formulated before for quantum superalgebras, we find it desirable to work out the formulas and constructions in detail in this rank one setting. The new features and connections observed in this paper will be instrumental in a forthcoming work [4] joint with David Hill on canonical basis for general quantum Kac-Moody superalgebras.

The algebra  $\dot{U}$  is modified from a quantum enveloping superalgebra  $U$  for  $\mathfrak{osp}(1|2)$  by adding idempotents, following [2, 12]. Our (Hopf) superalgebra  $U$  is defined as a direct sum of  $\mathbb{Q}(q)$ -superalgebras  $U_0$  and  $U_1$ , where  $U_0$  and  $U_1$  differ somewhat from the quantum  $\mathfrak{osp}(1|2)$  used in the literature (cf. [8, 1, 13, 7, 3]). In contrast to those variants, our algebras  $U_0, U_1$  and  $U$  are well suited for introducing a bar-involution and an integral form as needed in the construction of canonical basis, and the modified algebra  $\dot{U}$  has an intrinsic description. The bar-involution on  $U$  and  $\dot{U}$  used in this paper has the unusual feature that it sends a quantum parameter  $q$  to  $-q^{-1}$  (cf. [5]).

The complexified algebras  ${}^{\mathbb{C}}U_0$  for  $U_0$  and  ${}^{\mathbb{C}}U_1$  for  $U_1$  are shown to be isomorphic, and finite-dimensional simple modules of  ${}^{\mathbb{C}}U_0$  were classified in [13] in terms of highest weights labeled by pairs  $(n, \pm)$  for  $n \in \mathbb{N}$ . We show those even-weight (i.e., odd-dimensional) simple  ${}^{\mathbb{C}}U_0$ -modules arise from the simple  $U_0$ -modules while those odd-weight (i.e., even-dimensional) simple  ${}^{\mathbb{C}}U_0$ -modules arise from the simple  $U_1$ -modules.

Following [5], we introduce a covering quantum algebra  $U^\pi$  for  $\mathfrak{osp}(1|2)$  with an additional parameter  $\pi$  such that  $\pi^2 = 1$ . The covering algebra  $U^\pi$  admits a modified version  $\dot{U}^\pi$  too. The structure constants when multiplying the canonical basis elements in  $\dot{U}^\pi$  are

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positive integer Laurent polynomials in  $q$  and  $\pi$ . We expect that the algebra  $\dot{U}^\pi$  and its canonical basis can be categorified in a generalized framework of spin nilHecke algebras (à la Lauda [10] for  $\dot{U}_q(\mathfrak{sl}(2))$ ), where  $\pi$  again is categorified as a parity shift functor as in [5]. The algebras  $U^\pi$  and  $\dot{U}^\pi$  specialize when  $\pi = 1$  to  $U_q(\mathfrak{sl}(2))$  and its modified version, and specialize when  $\pi = -1$  to  $\dot{U}$  and  $\dot{U}^\pi$ . In particular, the canonical basis for  $\dot{U}^\pi$  are shown to specialize when  $\pi = 1$  and  $\pi = -1$  to the canonical basis for modified quantum  $\mathfrak{sl}(2)$  [12] and for  $\dot{U}$ , respectively. In other words, our constructions and formulas can be regarded as a  $\pi$ -enhanced version of their counterparts for quantum  $\mathfrak{sl}(2)$ .

It is well known that Lie superalgebra  $\mathfrak{osp}(1|2)$  admits only odd-dimensional simple modules. In contrast, the quantum  $\mathfrak{osp}(1|2)$  as defined in this paper has richer representation theory, which are compatible with the categorification construction and also with quantum  $\mathfrak{sl}(2)$ . All these will afford a natural generalization in the setting of quantum Kac-Moody superalgebras.

This paper is organized as follows. In Section 2, we define the algebras  $U_0, U_1$  and study their basic structures including the integral forms and (anti-)automorphisms. In Section 3, we classify the finite-dimensional simple weight modules of  $U_0$  and  $U_1$ . In Section 4, we show  $U = U_0 \oplus U_1$  has a natural Hopf superalgebra structure. In Section 5, we find an explicit formula for the quasi- $R$ -matrix of  $U$ , which is then used in defining the bar-involution for a tensor product of modules. The canonical basis on the tensor product of two finite-dimensional  $U$ -modules is computed. In Section 6, we define the modified algebra  $\dot{U}$ , compute its canonical basis, and formulate a bilinear form on  $\dot{U}$ . In Section 7, we formulate in the framework of covering algebras variants of constructions and results in the previous sections.

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## 2. STRUCTURES OF QUANTUM $\mathfrak{osp}(1|2)$

### 2.1. Algebra $U_0$ . Set

$$\pi = -1$$

throughout this paper except the final Section 7, and we will use the symbol  $\pi$  for the super signs in superalgebras arising from exchanges of odd elements. This allows us to state clean commutation formulas, and to recover many classical formulas for quantum  $\mathfrak{sl}(2)$  by simply dropping  $\pi$ .

**Definition 2.1.** *The algebra  $U_0$  is the  $\mathbb{Q}(q)$ -algebra generated by  $E, F, K$ , and  $K^{-1}$ , subject to the relations:*

- (1)  $KK^{-1} = 1 = K^{-1}K$ ;
- (2)  $KEK^{-1} = q^2E$ ,  $KFK^{-1} = q^{-2}F$ ;
- (3)  $EF - \pi FE = \frac{K - K^{-1}}{\pi q - q^{-1}}$ .

*Remark 2.2.* There has been definitions for quantum enveloping algebra of  $\mathfrak{osp}(1|2)$ , which differ from  $U_0$  by a different rescaling of the relation (3) above. A version of  $U_q(\mathfrak{osp}(1|2))$  appeared in [1, 13], where (3) is replaced by

$$(3a) \quad EF - \pi FE = \frac{K - K^{-1}}{q^2 - q^{-2}}.$$

On the other hand, the definition used in [7] replaces (3) by

$$(3b) \quad EF - \pi FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

These variants of  $U_q(\mathfrak{osp}(1|2))$  are all isomorphic to  $U_0$  as  $\mathbb{Q}(q)$ -algebras, with isomorphisms given by fixing  $F$  and  $K$ , and then by rescaling  $E$  by suitable scalars in  $\mathbb{Q}(q)$ . Our Definition 2.1 is most suitable for introducing an integral form  $\mathcal{A}U$  and a bar-involution  $\bar{\cdot} : U \rightarrow U$  below. As we shall see, (3b) is not bar-invariant under the bar-involution (2.6), while (3a) is not well suited for constructing an integral form.

**2.2. Algebra  $U_1$ .** We introduce a variant of quantum enveloping algebra for  $\mathfrak{osp}(1|2)$ .

**Definition 2.3.** *The algebra  $U_1$  is the  $\mathbb{Q}(q)$ -algebra generated by  $E, F, K$ , and  $K^{-1}$ , subject to the relations:*

- (1)  $KK^{-1} = 1 = K^{-1}K$ ;
- (2)  $KEK^{-1} = q^2E$ ,  $KFK^{-1} = q^{-2}F$ ;
- (3)  $EF - \pi FE = \frac{\pi K - K^{-1}}{\pi q - q^{-1}}$ .

Note the difference between definitions of  $U_0$  and  $U_1$  lies in the relation (3).

*Remark 2.4.* As we need to mix the use of  $U_0$  and  $U_1$ , we shall denote the generators for  $U_0$  (respectively,  $U_1$ ) by  $E_0, F_0, K_0$  (respectively,  $E_1, F_1, K_1$ ). Then the defining relations of  $U_\epsilon$  ( $\epsilon = 0, 1$ ) can be succinctly rewritten as

- (1)  $K_\epsilon K_\epsilon^{-1} = 1 = K_\epsilon^{-1} K_\epsilon$ ;
- (2)  $K_\epsilon E_\epsilon K_\epsilon^{-1} = q^2 E_\epsilon$ ,  $K_\epsilon F_\epsilon K_\epsilon^{-1} = q^{-2} F_\epsilon$ ;
- (3)  $E_\epsilon F_\epsilon - \pi F_\epsilon E_\epsilon = \frac{\pi^\epsilon K_\epsilon - K_\epsilon^{-1}}{\pi q - q^{-1}}$ .

The algebra  $U_\epsilon$  is naturally a superalgebra by letting  $E_\epsilon, F_\epsilon$  be odd and  $K_\epsilon^{\pm 1}$  be even.

**2.3. Complexification.** Fix a square root  $\sqrt{\pi} \in \mathbb{C}$ . For  $\epsilon = 0, 1$ , denote

$${}^{\mathbb{C}}U_\epsilon = \mathbb{C}(q) \otimes_{\mathbb{Q}(q)} U_\epsilon.$$

Though  $U_0$  and  $U_1$  are not isomorphic as  $\mathbb{Q}(q)$ -algebras, we have the following.

**Lemma 2.5.** *There is an isomorphism of  $\mathbb{C}(q)$ -algebras  $\flat : {}^{\mathbb{C}}U_1 \rightarrow {}^{\mathbb{C}}U_0$  such that*

$$\flat(F_1) = F_0, \quad \flat(E_1) = \sqrt{\pi}E_0, \quad \flat(K_1) = \sqrt{\pi}^{-1}K_0.$$

We may formally regard  $U_0$  and  $U_1$  as two different real forms for the same  $\mathbb{C}(q)$ -algebra. They share many of the same structural properties, and the proofs of these properties are quite similar. The rationale of introducing  $U_1$  besides  $U_0$  comes from Sections 3 and 6.

**2.4. PBW and gradings.** Clearly the elements  $F_\epsilon^a K_\epsilon^b E_\epsilon^c$  with  $a, c \in \mathbb{N}$  and  $b \in \mathbb{Z}$  span  $U_\epsilon$  since any monomial in  $E_\epsilon, F_\epsilon$ , and  $K_\epsilon$  can be expressed as a sum of such elements by using the defining relations. Proving linear independence can be done as in [6, 1.5]. Hence we obtain the following.

**Proposition 2.6.** *The algebra  $U_\epsilon$ , for  $\epsilon = 0, 1$ , has the following (PBW) bases:*

$$\left\{ F_\epsilon^a K_\epsilon^b E_\epsilon^c \mid a, c \in \mathbb{N}, b \in \mathbb{Z} \right\}, \quad \left\{ E_\epsilon^a K_\epsilon^b F_\epsilon^c \mid a, c \in \mathbb{N}, b \in \mathbb{Z} \right\}.$$

Let  $U_\epsilon^+$  be the subalgebra of  $U_\epsilon$  generated by  $E_\epsilon$ ,  $U_\epsilon^-$  be the subalgebra generated by  $F_\epsilon$ , and  $U_\epsilon^0$  be the subalgebra generated by  $K_\epsilon, K_\epsilon^{-1}$ .

The algebra  $U_\epsilon$  has two natural gradings on it: the  $\mathbb{Z}$ -grading arising from weight space decomposition of  $\mathfrak{osp}(1|2)$ , and a parity  $\mathbb{Z}_2$ -grading arising from the superalgebra structure of  $\mathfrak{osp}(1|2)$ . The *parity  $\mathbb{Z}_2$ -grading* on the algebra  $U_\epsilon$  is defined by

$$p(E_\epsilon) = p(F_\epsilon) = 1, \quad p(K_\epsilon) = p(K_\epsilon^{-1}) = 0.$$

The *weight  $\mathbb{Z}$ -grading* on the algebra  $U_\epsilon$  (which is the same as a weight space decomposition in our rank one setting) is defined by

$$|E_\epsilon| = 2, \quad |F_\epsilon| = -2, \quad |K_\epsilon| = |K_\epsilon^{-1}| = 0,$$

since the defining relations are clearly homogeneous with respect to this definition. We have

$$U_\epsilon = \bigoplus_{i \in 2\mathbb{Z}} U_\epsilon(i), \quad U_\epsilon(i) = \{u \in U_\epsilon | K_\epsilon u K_\epsilon^{-1} = q^i u\}.$$

**2.5. The  $\mathcal{A}$ -subalgebra.** Let

$$\mathcal{A} = \mathbb{Z}[q, q^{-1}], \quad \mathbb{N} = \{0, 1, 2, \dots\}.$$

For  $n \in \mathbb{Z}$  and  $a \in \mathbb{N}$ , we define the *super quantum integer* or  $(q, \pi)$ -integer

$$[n] = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}}, \quad (2.1)$$

and then define the corresponding factorials and binomial coefficients

$$[a]! = \prod_{i=1}^a [i], \quad \begin{bmatrix} n \\ a \end{bmatrix} = \frac{\prod_{i=1}^a [n+i-a]}{[a]!}. \quad (2.2)$$

We adopt the convention that  $[0]! = 1$ . Note that  $\begin{bmatrix} n \\ a \end{bmatrix} = \frac{[n]!}{[a]! [n-a]!}$ , for  $n \geq a \geq 0$ . One checks that  $[n] \in \mathcal{A}$ ,  $\begin{bmatrix} n \\ a \end{bmatrix} \in \mathcal{A}$ . A straightforward computation gives us

$$[-n] = -\pi^n [n], \quad \begin{bmatrix} n \\ a \end{bmatrix} = (-1)^a \pi^{na + \binom{a}{2}} \begin{bmatrix} a-n-1 \\ a \end{bmatrix}. \quad (2.3)$$

We use these super quantum integers to define the divided powers:

$$E_\epsilon^{(a)} = \frac{E_\epsilon^a}{[a]!}, \quad F_\epsilon^{(a)} = \frac{F_\epsilon^a}{[a]!}. \quad (2.4)$$

It is understood that  $E_\epsilon^{(0)} = F_\epsilon^{(0)} = 1$ . For  $n \in \mathbb{Z}, a \in \mathbb{N}$ , we also define the following elements in  $U_\epsilon$  (compare [6]):

$$[K_\epsilon; n] = \frac{(\pi q)^n \pi^\epsilon K_\epsilon - q^{-n} K_\epsilon^{-1}}{\pi q - q^{-1}}, \quad \begin{bmatrix} K_\epsilon; n \\ a \end{bmatrix} = \frac{\prod_{j=1}^a [K_\epsilon; n+j-a]}{[a]!}. \quad (2.5)$$

We let  ${}_{\mathcal{A}}U_\epsilon$  be the  $\mathcal{A}$ -subalgebra of  $U_\epsilon$  generated by  $E_\epsilon^{(a)}, F_\epsilon^{(a)}, K_\epsilon^{\pm 1}, \begin{bmatrix} K_\epsilon; n \\ a \end{bmatrix}$ , for  $n \in \mathbb{Z}, a \in \mathbb{N}$ .

**2.6. Automorphisms.** Following a key observation in [5], we define the  $\mathbb{Q}$ -automorphism of  $\mathbb{Q}(q)$ , denoted by  $\bar{\phantom{x}}$ , such that

$$\bar{q} = \pi q^{-1}. \quad (2.6)$$

Note that the super quantum integers are bar-invariant. A map  $\phi$  from a  $\mathbb{Q}(q)$ -algebra  $A$  to itself is called *antilinear* if  $\phi(g(q)a) = \overline{g(q)}\phi(a)$ , for  $g(q) \in \mathbb{Q}(q)$ . We also adopt the convention that an *anti-homomorphism*  $f$  on  $A$  is a  $\mathbb{Q}(q)$ -linear map satisfying  $f(xy) = f(y)f(x)$ , for  $x, y \in A$ . Below we shall denote by  $D_4$  the dihedral group of order 8.

**Proposition 2.7.** *Let  $\epsilon \in \{0, 1\}$ .*

- (1) *There is a  $\mathbb{Q}(q)$ -antilinear involution  $\psi_\epsilon : U_\epsilon \rightarrow U_\epsilon$  such that*

$$\psi_\epsilon(E_\epsilon) = E_\epsilon, \quad \psi_\epsilon(F_\epsilon) = F_\epsilon, \quad \psi_\epsilon(K_\epsilon) = \pi^\epsilon K_\epsilon^{-1};$$

*( $\psi_\epsilon$  is referred to as the bar involution and also denoted by  $\bar{\phantom{x}} : U_\epsilon \rightarrow U_\epsilon$ ).*

- (2) *There is a  $\mathbb{Q}(q)$ -linear automorphism  $\omega_\epsilon : U_\epsilon \rightarrow U_\epsilon$  such that*

$$\omega_\epsilon(E_\epsilon) = F_\epsilon, \quad \omega_\epsilon(F_\epsilon) = \pi^{1-\epsilon} E_\epsilon, \quad \omega_\epsilon(K_\epsilon) = K_\epsilon^{-1};$$

- (3) *There is a  $\mathbb{Q}(q)$ -linear anti-involution  $\tau_\epsilon : U_\epsilon \rightarrow U_\epsilon$  such that*

$$\tau_\epsilon(E_\epsilon) = \pi^{1-\epsilon} E_\epsilon, \quad \tau_\epsilon(F_\epsilon) = F_\epsilon, \quad \tau_\epsilon(K_\epsilon) = K_\epsilon^{-1};$$

- (4) *There is a  $\mathbb{Q}(q)$ -linear anti-involution  $\rho_\epsilon : U_\epsilon \rightarrow U_\epsilon$  such that*

$$\rho_\epsilon(E_\epsilon) = qK_\epsilon F_\epsilon, \quad \rho_\epsilon(F_\epsilon) = qK_\epsilon^{-1} E_\epsilon, \quad \rho_\epsilon(K_\epsilon) = K_\epsilon.$$

- (5) *The subgroup of (anti-)automorphisms on  $U_\epsilon$  generated by  $\omega_\epsilon, \tau_\epsilon, \psi_\epsilon$  is isomorphic to  $D_4 \times \mathbb{Z}_2$  for  $\epsilon = 0$  and to  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$  for  $\epsilon = 1$ . More precisely,*

$$\begin{aligned} \omega_0^4 &= 1, \quad \omega_1^2 = 1, \quad \tau_0 \omega_0 = \omega_0^3 \tau_0, \quad \tau_1 \omega_1 = \omega_1 \tau_1, \\ \tau_\epsilon^2 &= \psi_\epsilon^2 = 1, \quad \psi_\epsilon \tau_\epsilon = \tau_\epsilon \psi_\epsilon, \quad \psi_\epsilon \omega_\epsilon = \omega_\epsilon \psi_\epsilon. \end{aligned}$$

*Proof.* This is proved by a direct computation, and let us suppress the subscript  $\epsilon$ . To illustrate, let us verify that the (most involved) commutation relation (3) in Remark 2.4 between  $E$  and  $F$  is preserved under these maps. Since  $\psi$  fixes  $E, F$ , and  $\pi^\epsilon K - K^{-1}$ , it preserves the relation between  $E$  and  $F$ , whence (1).

To verify for (2), we compute

$$\begin{aligned} \omega(EF - \pi FE) &= \pi^{1-\epsilon} FE - \pi^\epsilon EF = -\pi^\epsilon (EF - \pi FE), \\ \omega\left(\frac{\pi^\epsilon K - K^{-1}}{\pi q - q^{-1}}\right) &= -\pi^\epsilon \left(\frac{\pi^\epsilon K - K^{-1}}{\pi q - q^{-1}}\right). \end{aligned}$$

For (4), we further compute

$$\begin{aligned} \rho(EF - \pi FE) &= q^2 K^{-1} E K F - \pi q^2 K F K^{-1} E = EF - \pi FE, \\ \rho\left(\frac{\pi^\epsilon K - K^{-1}}{\pi q - q^{-1}}\right) &= \frac{\pi^\epsilon K - K^{-1}}{\pi q - q^{-1}}. \end{aligned}$$

The calculation for  $\tau$  in (3) is exactly the same as for  $\omega$ . Finally (5) may be quickly verified by checking on the generators.  $\square$

Let  $\epsilon \in \{0, 1\}$ . By Proposition 2.7, we have the following identities in  $U_\epsilon$ : for  $n \in \mathbb{Z}, a \in \mathbb{N}$ ,

$$\begin{aligned}\omega_\epsilon(E_\epsilon^{(r)}) &= F_\epsilon^{(r)}, & \omega_\epsilon(F_\epsilon^{(r)}) &= \pi^{r(1-\epsilon)} E_\epsilon^{(r)}, \\ \omega_\epsilon([K_\epsilon; n]) &= -\pi^{\epsilon+n} [K_\epsilon; -n], \\ \omega_\epsilon\left(\begin{bmatrix} K_\epsilon; n \\ a \end{bmatrix}\right) &= (-1)^a \pi^{\epsilon a + na - \binom{a}{2}} \begin{bmatrix} K_\epsilon; a - n - 1 \\ a \end{bmatrix}.\end{aligned}\tag{2.7}$$

It is straightforward to check the following identities in  ${}_{\mathcal{A}}U_\epsilon$ : for  $a, b, c, s \in \mathbb{Z}$ ,

$$\begin{aligned}[b + c][K_\epsilon; a] &= [b][K_\epsilon; a + c] + \pi^b [c][K_\epsilon; a - b], \\ E_\epsilon[K_\epsilon; s] &= [K_\epsilon; s - 2]E_\epsilon, \\ F_\epsilon[K_\epsilon; s] &= [K_\epsilon; s + 2]F_\epsilon.\end{aligned}\tag{2.8}$$

## 2.7. Commutation relations.

**Lemma 2.8.** *Let  $\epsilon \in \{0, 1\}$ . The following identities hold in  ${}_{\mathcal{A}}U_\epsilon$ : for  $r, s \geq 1$ ,*

$$\begin{aligned}(1) \quad & \pi^s E_\epsilon F_\epsilon^{(s)} = F_\epsilon^{(s)} E_\epsilon + \pi F_\epsilon^{(s-1)} [K_\epsilon; 1 - s]; \\ (2) \quad & \pi^{rs} E_\epsilon^{(r)} F_\epsilon^{(s)} = \sum_{i=0}^{\min(r,s)} \pi^{\binom{i+1}{2}} F_\epsilon^{(s-i)} \begin{bmatrix} K_\epsilon; 2i - (r + s) \\ i \end{bmatrix} E_\epsilon^{(r-i)}; \\ (3) \quad & \pi^s F_\epsilon E_\epsilon^{(s)} = E_\epsilon^{(s)} F_\epsilon - \pi^{1-s} E_\epsilon^{(s-1)} [K_\epsilon; s - 1]; \\ (4) \quad & \pi^{rs} F_\epsilon^{(s)} E_\epsilon^{(r)} = \sum_{i=0}^{\min(r,s)} (-1)^i \pi^{i(r+s)} E_\epsilon^{(r-i)} \begin{bmatrix} K_\epsilon; r + s - (i + 1) \\ i \end{bmatrix} F_\epsilon^{(s-i)}.\end{aligned}$$

*Proof.* The first two identities (1) and (2) can be proven using induction. Fix  $\epsilon \in \{0, 1\}$ . Again, we suppress the subscripts throughout the proof.

(1). The base case  $s = 1$  is a defining relation for  $U_\epsilon$ . Now suppose that the identity (1) holds for some  $s$ . Then

$$\begin{aligned}\pi^{s+1} E F^{(s)} F &= \pi F^{(s)} E F + \pi^2 F^{(s-1)} [K; 1 - s] F \\ &= F^{(s)} F E + \pi F^{(s)} [K; 0] + \pi^2 [s] F^{(s)} [K; -1 - s] \\ &= F^{(s)} F E + \pi F^{(s)} ([K; 0] + \pi [s] [K; -1 - s]) \\ &= F^{(s)} F E + \pi F^{(s)} [s + 1] [K; -s]\end{aligned}$$

The last equality follows from (2.8) with  $a = -s$ ,  $b = 1$ , and  $c = s$ . Dividing both sides by  $[s + 1]$  finishes the induction step.

(2). We proceed by induction on  $r$ , with the case case for  $r = 1$  being (1). Suppose now that the identity (2) holds for some  $r$ . Then

$$\begin{aligned}
\pi^{rs+s} E E^{(r)} F^{(s)} &= \sum_{i=0}^{\min(r,s)} \pi^{\binom{i+1}{2}} \pi^s E F^{(s-i)} \begin{bmatrix} K; 2i - (r+s) \\ i \end{bmatrix} E^{(r-i)} \\
&= \sum_{i=0}^{\min(r,s)} \pi^{\binom{i+1}{2}} \pi^i F^{(s-i)} E \begin{bmatrix} K; 2i - (r+s) \\ i \end{bmatrix} E^{(r-i)} \\
&\quad + \sum_{i=0}^{\min(r,s)} \pi^{\binom{i+1}{2}} \pi^{i+1} F^{(s-i-1)} [K; 1+i-s] \begin{bmatrix} K; 2i - (r+s) \\ i \end{bmatrix} E^{(r-i)} \\
&= \sum_{i=0}^{\min(r,s)} \pi^{\binom{i+1}{2}} \pi^i F^{(s-i)} \begin{bmatrix} K; 2i - (r+s+2) \\ i \end{bmatrix} E E^{(r-i)} \\
&\quad + \sum_{i=1}^{\min(r,s)+1} \pi^{\binom{i}{2}} \pi^i F^{(s-i)} [K; i-s] \begin{bmatrix} K; 2i - (r+s+2) \\ i-1 \end{bmatrix} E^{(r-i+1)} \\
&= \sum_{i=0}^{\min(r+1,s)} \pi^{\binom{i+1}{2}} F^{(s-i)} X_i E^{(r+1-i)}. \tag{2.9}
\end{aligned}$$

Here  $X_0 = [r+1]$ ,  $X_{r+1} = [r+1] \begin{bmatrix} K; r+1-s \\ r+1 \end{bmatrix}$  if  $r < s$ , and for  $1 \leq i \leq \min(r, s)$ ,

$$\begin{aligned}
X_i &= \pi^i [r+1-i] \begin{bmatrix} K; 2i - (r+s+2) \\ i \end{bmatrix} + [K; i-s] \begin{bmatrix} K; 2i - (r+s+2) \\ i-1 \end{bmatrix} \\
&= [i]^{-1} \begin{bmatrix} K; 2i - (r+s+2) \\ i-1 \end{bmatrix} (\pi^i [r+1-i] [K; i - (r+s+1)] + [i] [K; i-s]) \\
&\stackrel{(*)}{=} [i]^{-1} \begin{bmatrix} K; 2i - (r+s+2) \\ i-1 \end{bmatrix} [r+1] [K; 2i - (r+s+1)] \\
&= [r+1] \begin{bmatrix} K; 2i - (r+s+1) \\ i \end{bmatrix}.
\end{aligned}$$

The equality (\*) above follows from (2.8) with  $a = 2i - (r+s+1)$ ,  $b = i$ , and  $c = r+1-i$ . Dividing both sides of (2.9) by  $[r+1]$  we obtain (2).

The identities (3) and (4) follow by applying the automorphism  $\omega_\epsilon$  to (1) and (2) and using (2.7).  $\square$

### 3. FINITE-DIMENSIONAL REPRESENTATIONS

**3.1. Weight  $U_\epsilon$ -modules.** Let us now turn to  $U_\epsilon$ -modules, for  $\epsilon = 1, 2$ . We will call a  $U_\epsilon$ -module  $M$  a *weight module* if the action of  $K$  on  $M$  is semisimple with finite-dimensional eigenspaces (i.e., weight spaces). The *Verma module* of  $U_\epsilon$  of highest weight  $\lambda \in \mathbb{Q}(q)$  is defined to be

$$M_\epsilon^\lambda = U_\epsilon / (U_\epsilon E_\epsilon + U_\epsilon (K_\epsilon - \lambda)),$$

with an even highest weight vector denoted by  $\nu$ . Then by Proposition 2.6  $M_\epsilon^\lambda$  has a basis given by  $F_\epsilon^{(k)} \nu$ , for  $k \geq 0$ . Denote by  $L_\epsilon^\lambda$  for now the unique irreducible quotient module of  $M_\epsilon^\lambda$ . We observe the following three statements are equivalent: (i) The  $U_\epsilon$ -module  $M_\epsilon^\lambda$

is reducible; (2)  $M_\epsilon^\lambda$  admits a (singular) vector  $F^{(t)}\nu$  for some  $t > 0$  annihilate by  $E_\epsilon$ ; (3)  $L_\epsilon^\lambda$  is finite dimensional. By Lemma 2.8, we have

$$E_\epsilon F_\epsilon^{(t)}\nu = \pi F_\epsilon^{(t-1)}[K_\epsilon; 1-t]\nu.$$

A quick calculation using this equation to locate a possible singular vector in  $M_\epsilon^\lambda$  leads to the following.

**Proposition 3.1.** *Let  $\epsilon \in \{0, 1\}$ .*

- (1)  $M_\epsilon^\lambda$  is an irreducible  $U_\epsilon$ -module, unless  $\lambda = \pm q^n$  for  $n \in \epsilon + 2\mathbb{N}$ .
- (2) For each  $n \in \epsilon + 2\mathbb{N}$ , there is a unique pair of  $(n+1)$ -dimensional simple  $U_\epsilon$ -modules  $L(n, \pm) := L_\epsilon^{\pm q^n}$  of highest weight  $\pm q^n$ . Moreover, any finite-dimensional simple weight  $U_\epsilon$ -module is isomorphic to one such module.

This result should be compared to the classification of finite-dimensional simple modules for  ${}^{\mathbb{C}}U_0$  below.

**Proposition 3.2.** [13] *For each  $n \in \mathbb{N}$ , there are two non-isomorphic  $(n+1)$ -dimensional  ${}^{\mathbb{C}}U_0$ -modules over  $\mathbb{C}(q)$  of highest weight  $\pi^{n^2/2}q^n$ . Moreover, any finite-dimensional  ${}^{\mathbb{C}}U_0$ -module is completely reducible.*

*Remark 3.3.* Note that the weights of the simple  ${}^{\mathbb{C}}U_0$ -modules for  $n$  odd in Proposition 3.2 involve complex number  $\sqrt{\pi}$ , and so they cannot be realized as  $U_0$ -modules over  $\mathbb{Q}(q)$ . This partially motivated our introduction of  $U_1$ .

*Remark 3.4.* Proposition 3.2 remains to be valid if we classify finite-dimensional modules of  $Q[\sqrt{\pi}](q) \otimes_{\mathbb{Q}(q)} U_0$  over the field  $\mathbb{Q}[\sqrt{\pi}](q)$  instead of  $\mathbb{C}(q)$ .

Note that the “weight”  $U_\epsilon$ -module condition in Proposition 3.1 is necessary over  $\mathbb{Q}(q)$ . Indeed, if we view the  $Q[\sqrt{\pi}](q)$ -vector space underlying a 2-dimensional module of  $Q[\sqrt{\pi}](q) \otimes_{\mathbb{Q}(q)} U_0$  as a  $\mathbb{Q}(q)$ -vector space, we obtain a 4-dimensional  $U_0$ -module which is not a weight module.

**3.2. Complete reducibility.** It has been known that there is a Casimir element for (a version of) the algebra  $U_0$  (see e.g. [1]). Let  $\epsilon \in \{0, 1\}$ . We adapt this construction to the algebras  $U_\epsilon$ . We will proceed as in [6, §§2.7-2.9]. Set

$$C_\epsilon = \pi F_\epsilon E_\epsilon + \frac{\pi^{1-\epsilon} K_\epsilon q + K_\epsilon^{-1} q^{-1}}{(\pi q - q^{-1})^2}. \quad (3.1)$$

One rewrites using defining relations of  $U_\epsilon$  that

$$C_\epsilon = E_\epsilon F_\epsilon + \frac{\pi^\epsilon K_\epsilon q^{-1} + \pi K_\epsilon^{-1} q}{(\pi q - q^{-1})^2}.$$

We note that  $\omega_\epsilon(C_\epsilon) = \tau_\epsilon(C_\epsilon) = \pi^\epsilon C_\epsilon$ . Also, we have that

$$C_\epsilon E_\epsilon = \pi E_\epsilon C_\epsilon, \quad C_\epsilon F_\epsilon = \pi F_\epsilon C_\epsilon, \quad C_\epsilon K_\epsilon = K_\epsilon C_\epsilon. \quad (3.2)$$

Indeed, clearly we have  $C_\epsilon K_\epsilon = K_\epsilon C_\epsilon$ . We compute

$$\begin{aligned} C_\epsilon E_\epsilon &= E_\epsilon F_\epsilon E_\epsilon + \frac{\pi^\epsilon K_\epsilon q^{-1} + \pi K_\epsilon^{-1} q}{(\pi q - q^{-1})^2} E_\epsilon \\ &= \pi \left( E_\epsilon F_\epsilon E_\epsilon + E_\epsilon \frac{\pi^\epsilon K_\epsilon q + K_\epsilon^{-1} q^{-1}}{(\pi q - q^{-1})^2} \right) = \pi E_\epsilon C_\epsilon. \end{aligned}$$

The remaining identity in (3.2) can be checked similarly. It follows by (3.2) that  $C_\epsilon^2$  is in the center of  $U_\epsilon$ .



**Proposition 3.5.** *Let  $\epsilon \in \{0, 1\}$  and  $n \in \mathbb{Z}$ . Then,*

- (1)  $C_\epsilon^2$  acts on the Verma module  $M_\epsilon^{\pm q^n}$  as scalar multiplication by  $\frac{[n+1]^2}{(\pi q - q^{-1})^2}$ .
- (2)  $C_\epsilon^2$  acts on  $M_\epsilon^{\pm q^n}$  and  $M_\epsilon^{\pm q^m}$  by the same scalar if and only if  $n = m$  or  $n = -m - 2 \in \mathbb{Z}$ ; in particular,  $C_\epsilon^2$  acts as a different scalar on different pairs  $L(n, \pm)$ , for  $n \in \epsilon + 2\mathbb{Z}_+$ .
- (3) Any finite-dimensional weight  $U_\epsilon$ -module is completely reducible.

*Proof.* Let  $\nu$  be the highest weight vector of  $\Lambda_n$ . Using (3.1), we see that  $C_\epsilon^2 \nu = \frac{[n+1]^2}{(\pi q - q^{-1})^2} \nu$ . Since any  $m \in M_\epsilon^{\pm q^n}$  can be represented as  $m = u\nu$  for  $u \in U_\epsilon$ ,  $C_\epsilon^2 m = C_\epsilon^2 u\nu = uC_\epsilon^2 \nu = (\pi q - q^{-1})^{-2} [n+1]^2 m$ , whence (1).

Now  $[n+1]^2 = [m+1]^2$  if and only if  $(\pi q)^{n+1} - q^{-n-1} = \pm((\pi q)^{m+1} - q^{-m-1})$ , whence (2). For a given  $n \in \mathbb{Z}_+$ , by weight considerations there is no nontrivial extension between  $L(n, +)$  and  $L(n, -)$ . We can prove (3) as is done in [6, §2.9]; that is, pick a composition series for  $M$  and use a weight dimension argument to show that composition factors are direct summands.  $\square$

#### 4. THE HOPF SUPERALGEBRA $U$

**4.1. Algebra  $U$ .** By the similarities of  $U_\epsilon$  and  $U_q(\mathfrak{sl}(2))$ , we hope to make sense that the tensor product of two odd-weight modules should decompose as a sum of even-weight modules. It is therefore convenient to combine  $U_0$  and  $U_1$  into a single algebra.

**Definition 4.1.** *The algebra  $U$  is defined to be the direct sum of algebras  $U = U_0 \oplus U_1$ , whose multiplication is denoted by  $m$ . Let  $e_0 = (1, 0)$  and  $e_1 = (0, 1)$  be the central idempotents of  $U$  with  $U_0 = e_0 U$ ,  $U_1 = e_1 U$  and  $e_0 e_1 = 0$ ; hence  $U$  is a unital algebra with  $1 = e_0 + e_1$ .*

Another possible way is to define a smaller single algebra so that both  $U_0$  and  $U_1$  become the quotient algebras, but we will not follow that route in this paper.

It is immediate that the direct sums (over  $\epsilon = 0, 1$ ) of the (anti-)automorphisms  $\psi_\epsilon$ ,  $\omega_\epsilon$ ,  $\tau_\epsilon$ , and  $\rho_\epsilon$  define (anti-)automorphisms  $\psi$ ,  $\omega$ ,  $\tau$ , and  $\rho$  on  $U$ , respectively. We also have the  $\mathcal{A}$ -subalgebra  $\mathcal{A}U = \mathcal{A}U_0 \oplus \mathcal{A}U_1$  and a  $\mathbb{Z}$ -grading  $U = \bigoplus_{i \in 2\mathbb{Z}} U(i)$ , where  $U(i) = U_0(i) + U_1(i)$ . Since  $U$  is a direct sum of unital algebras, each  $U$ -module  $M$  decomposes as  $M = M_0 \oplus M_1$  where  $M_\epsilon = e_\epsilon M$  is a  $U_\epsilon$ -module ( $\epsilon = 0, 1$ ), and  $U_1 M_0 = U_0 M_1 = 0$ . We shall call a  $U$ -module  $M = M_0 \oplus M_1$  a weight module if  $M_0$  and  $M_1$  are weight modules. We may restate Proposition 3.1 and Proposition 3.5(3) in a form more commensurate with Proposition 3.2 and also with representation theory of  $U_q(\mathfrak{sl}(2))$  ([6]).

**Proposition 4.2.** *For each  $n \in \mathbb{N}$ , there is a pair of non-isomorphic  $(n+1)$ -dimensional simple  $U$ -modules denoted by  $L(n, \pm)$  of highest weight  $\pm q^n$ . Any finite dimensional simple weight  $U$ -module is isomorphic to one such module. Moreover, any finite-dimensional weight  $U$ -module is completely reducible.*

We will from now on concentrate only on  $L(n) := L(n, +)$ , since the cases of  $L(n, -)$  is completely parallel.

**4.2. Algebra  $\mathbf{f}$ .** Following Lusztig ([12]), there is a free  $\mathbb{Q}(q)$ -algebra  $\mathbf{f} = \mathbb{Q}(q)[\theta]$ , where  $\theta$  has  $\mathbb{Z}$ -grading 2 and parity  $p(\theta) = 1$ . We have natural  $\mathbb{Q}(q)$ -algebra isomorphisms  $(\cdot)_\epsilon^\pm : \mathbf{f} \rightarrow U_\epsilon^\pm$  given by  $\theta \mapsto \theta_\epsilon^+ = E_\epsilon$  and  $\theta \mapsto \theta_\epsilon^- = F_\epsilon$ . We define the maps  $(\cdot)^\pm : \mathbf{f} \rightarrow U$  by  $u^\pm = u_0^\pm \oplus u_1^\pm$ ; that is, it is the diagonal embedding  $\theta^+ = E_0 + E_1$  and  $\theta^- = F_0 + F_1$ .

We can define a bilinear form on  $\mathbf{f}$  such that

$$(\theta, \theta) = (1 - \pi q^{-2})^{-1}, \quad (4.1)$$

$$(\theta^{(a)}, \theta^{(b)}) = \delta_{a,b} \prod_{s=1}^a \frac{\pi^{s-1}}{1 - (\pi q^{-2})^s} = \delta_{a,b} \pi^a q^{\binom{a+1}{2}} (\pi q - q^{-1})^{-a} ([a]!)^{-1}. \quad (4.2)$$

A version of this bilinear form was first introduced in [5] for quantum Kac-Moody superalgebras including  $\mathfrak{osp}(1|2)$ , with a switch of  $q$  with  $q^{-1}$  in (4.1).

**4.3. The coproduct.** We endow the tensor product of superalgebras with the twisted multiplication

$$(a \otimes b) * (c \otimes d) = \pi^{p(b)p(c)} ac \otimes bd.$$

It is known that  $U_0$  is a Hopf superalgebra (cf. [13]). The following lemma can be regarded as an extension of the coproduct on  $U_0$  (compare [12, 3.1.3]).

**Lemma 4.3.** *For fixed  $\epsilon, \kappa \in \{0, 1\}$ , there is a unique (super)algebra homomorphism  $\Delta_{\epsilon, \kappa} : U_{\epsilon+\kappa} \rightarrow U_{\epsilon} \otimes U_{\kappa}$  satisfying*

$$\Delta_{\epsilon, \kappa}(E_{\epsilon+\kappa}) = E_{\epsilon} \otimes e_{\kappa} + \pi^{\epsilon} K_{\epsilon} \otimes E_{\kappa},$$

$$\Delta_{\epsilon, \kappa}(F_{\epsilon+\kappa}) = F_{\epsilon} \otimes K_{\kappa}^{-1} + e_{\epsilon} \otimes F_{\kappa},$$

$$\Delta_{\epsilon, \kappa}(K_{\epsilon+\kappa}) = K_{\epsilon} \otimes K_{\kappa}.$$

*Proof.* In the following, we shall suppress the subscripts on elements of  $U_{\epsilon+\kappa}$  since they are clear from context. We need to prove that the defining relations of  $U_{\epsilon+\kappa}$  are preserved by  $\Delta_{\epsilon, \kappa}$ . We will only check the most involved case as follows:

$$\begin{aligned} & \Delta_{\epsilon, \kappa}(E) \Delta_{\epsilon, \kappa}(F) - \pi \Delta_{\epsilon, \kappa}(F) \Delta_{\epsilon, \kappa}(E) \\ &= [E_{\epsilon}, F_{\epsilon}] \otimes K_{\kappa}^{-1} + \pi^{\epsilon} K_{\epsilon} \otimes [E_{\kappa}, F_{\kappa}] \\ &= \frac{(\pi^{\epsilon} K_{\epsilon} - K_{\epsilon}^{-1}) \otimes K_{\kappa}^{-1} + \pi^{\epsilon} K_{\epsilon} \otimes (\pi^{\kappa} K_{\kappa} - K_{\kappa}^{-1})}{\pi q - q^{-1}} \\ &= \frac{\pi^{\epsilon+\kappa} K_{\epsilon} \otimes K_{\kappa} - K_{\epsilon}^{-1} \otimes K_{\kappa}^{-1}}{\pi q - q^{-1}} = \Delta_{\epsilon, \kappa} \left( \frac{\pi^{\epsilon+\kappa} K - K^{-1}}{\pi q - q^{-1}} \right). \end{aligned}$$

The lemma is proved.  $\square$

**Lemma 4.4.** *The maps  $\Delta_{\epsilon, \kappa}$  are coassociative, that is, for  $\epsilon, \kappa, \iota \in \{0, 1\}$ , the following diagram is commutative:*

$$\begin{array}{ccc} U_{\epsilon+\kappa+\iota} & \xrightarrow{\Delta_{\epsilon, \kappa+\iota}} & U_{\epsilon} \otimes U_{\kappa+\iota} \\ \Delta_{\epsilon+\kappa, \iota} \downarrow & & \downarrow \text{id} \otimes \Delta_{\kappa, \iota} \\ U_{\epsilon+\kappa} \otimes U_{\iota} & \xrightarrow{\Delta_{\epsilon, \kappa} \otimes \text{id}} & U_{\epsilon} \otimes U_{\kappa} \otimes U_{\iota} \end{array}$$

*Proof.* We shall suppress subscripts on elements in  $U_{\epsilon+\kappa+\iota}$ . It suffices to check the commutativity on the generators; it is trivially true on  $K$ . We compute

$$\begin{aligned} & (\text{id} \otimes \Delta_{\kappa, \iota}) \circ \Delta_{\epsilon, \kappa+\iota}(F) = F_{\epsilon} \otimes K_{\kappa}^{-1} \otimes K_{\iota}^{-1} + e_{\epsilon} \otimes F_{\kappa} \otimes K_{\iota}^{-1} + e_{\epsilon} \otimes e_{\kappa} \otimes F_{\iota}, \\ & (\Delta_{\epsilon, \kappa} \otimes \text{id}) \circ \Delta_{\epsilon+\kappa, \iota}(F) = F_{\epsilon} \otimes K_{\kappa}^{-1} \otimes K_{\iota}^{-1} + e_{\epsilon} \otimes F_{\kappa} \otimes K_{\iota}^{-1} + e_{\epsilon} \otimes e_{\kappa} \otimes F_{\iota}, \\ & (\text{id} \otimes \Delta_{\kappa, \iota}) \circ \Delta_{\epsilon, \kappa+\iota}(E) = E_{\epsilon} \otimes e_{\kappa} \otimes e_{\iota} + \pi^{\epsilon} K_{\epsilon} \otimes E_{\kappa} \otimes e_{\iota} + \pi^{\epsilon+\kappa} K_{\epsilon} \otimes K_{\kappa} \otimes E_{\iota}, \\ & (\Delta_{\epsilon, \kappa} \otimes \text{id}) \circ \Delta_{\epsilon+\kappa, \iota}(E) = E_{\epsilon} \otimes e_{\kappa} \otimes e_{\iota} + \pi^{\epsilon} K_{\epsilon} \otimes E_{\kappa} \otimes e_{\iota} + \pi^{\epsilon+\kappa} K_{\epsilon} \otimes K_{\kappa} \otimes E_{\iota}. \end{aligned}$$

The lemma is proved.  $\square$

**Proposition 4.5.** *The superalgebra  $U$  endowed with the additional structures below is a Hopf superalgebra:*

- (1) a coproduct  $\Delta : U \rightarrow U \otimes U$  defined by  $\Delta = (\Delta_{0,0} + \Delta_{1,1}) \oplus (\Delta_{0,1} + \Delta_{1,0})$ ;
- (2) a counit  $\varepsilon : U \rightarrow \mathbb{Q}(q)$  defined by  $\varepsilon(e_1) = \varepsilon(E_0) = \varepsilon(F_0) = 0$  and  $\varepsilon(K_0) = 1$ ;
- (3) an antipode  $S : U \rightarrow U$  defined by  $S(K_\epsilon) = K_\epsilon^{-1}$ ,  $S(F_\epsilon) = -F_\epsilon K_\epsilon$  and  $S(E_\epsilon) = -\pi^\epsilon K_\epsilon^{-1} E_\epsilon$ , for  $\epsilon = 0, 1$ .

*Proof.* The statements on properties of  $\Delta$  are simply a reformulation of Lemmas 4.3 and 4.4. It is trivial to verify that the counit is indeed an algebra homomorphism and satisfies the defining commutative diagram for a counit; for example, to check that  $(\varepsilon \otimes 1) \circ \Delta(E_1) = 1 \otimes E_1$ , we compute

$$(\varepsilon \otimes 1) \circ \Delta(E_1) = \varepsilon(E_0) \otimes e_1 + \varepsilon(E_1) \otimes e_0 + \varepsilon(K_0) \otimes E_1 + \pi \varepsilon(K_1) \otimes e_0 = 1 \otimes E_1.$$

To show that the antipode is an anti-automorphism, it is trivial to check all except for the commutator relation between  $E_\epsilon$  and  $F_\epsilon$ , which we compute directly:

$$\begin{aligned} S(E_\epsilon F_\epsilon - \pi F_\epsilon E_\epsilon) &= \pi S(F_\epsilon) S(E_\epsilon) - \pi^2 S(E_\epsilon) S(F_\epsilon) \\ &= \pi \pi^\epsilon F_\epsilon E_\epsilon - \pi^\epsilon K_\epsilon E_\epsilon F_\epsilon K_\epsilon^{-1} = -\pi^\epsilon (E_\epsilon F_\epsilon - \pi F_\epsilon E_\epsilon) \\ &= \frac{\pi^\epsilon K^{-1} - K}{\pi q - q^{-1}} = S\left(\frac{\pi^\epsilon K_\epsilon - K_\epsilon^{-1}}{\pi q - q^{-1}}\right). \end{aligned}$$

Then we need to check that  $m \circ (S \otimes 1) \circ \Delta = m \circ (1 \otimes S) \circ \Delta = \iota \circ \varepsilon$  on the generators, where  $\iota : \mathbb{Q}(q) \rightarrow U$  is the  $\mathbb{Q}(q)$ -linear embedding sending  $1 \mapsto 1$ . This is trivial to check on  $E_1$ ,  $F_1$  and  $K_1$  since  $U_0 \otimes U_1 \oplus U_1 \otimes U_0$  is in the kernel of  $m$ . Checking this equality on  $E_0$ ,  $F_0$ , and  $K_0$  is essentially the same as the  $U_q(\mathfrak{sl}(2))$ -argument; for example,

$$m \circ (S \otimes 1) \circ \Delta(K_0) = S(K_0) K_0 + S(K_1) K_1 = e_0 + e_1 = 1 = \varepsilon(K_0).$$

The proposition is proved.  $\square$

The following is a super analogue of [12, 3.1.5].

**Lemma 4.6.** *The following formulas hold for  $\Delta : U \rightarrow U \otimes U$  and  $\epsilon = 0, 1$ :*

$$\begin{aligned} \Delta(E_0^{(p)}) &= \sum_{a+b=p} q^{ab} E_0^{(a)} K_0^b \otimes E_0^{(b)} + \sum_{a+b=p} \pi^b q^{ab} E_1^{(a)} K_1^b \otimes E_1^{(b)}, \\ \Delta(E_1^{(p)}) &= \sum_{a+b=p} q^{ab} E_0^{(a)} K_0^b \otimes E_1^{(b)} + \sum_{a+b=p} \pi^b q^{ab} E_1^{(a)} K_1^b \otimes E_0^{(b)}, \\ \Delta(F_0^{(p)}) &= \sum_{a+b=p} (\pi q)^{-ab} F_0^{(a)} \otimes K_0^{-a} F_0^{(b)} + \sum_{a+b=p} (\pi q)^{-ab} F_1^{(a)} \otimes K_1^{-a} F_1^{(b)}, \\ \Delta(F_1^{(p)}) &= \sum_{a+b=p} (\pi q)^{-ab} F_0^{(a)} \otimes K_1^{-a} F_1^{(b)} + \sum_{a+b=p} (\pi q)^{-ab} F_1^{(a)} \otimes K_0^{-a} F_0^{(b)}. \end{aligned}$$

*Proof.* The proof of all the four identities are similar, and we will only give the detail on the first one. To prove the first identity, it is equivalent to prove that

$$\begin{aligned} \Delta_{0,0}(E_0^{(p)}) &= \sum_{a+b=p} q^{ab} E_0^{(a)} K_0^b \otimes E_0^{(b)}, \\ \Delta_{1,1}(E_0^{(p)}) &= \sum_{a+b=p} \pi^b q^{ab} E_1^{(a)} K_1^b \otimes E_1^{(b)}. \end{aligned}$$

Let us verify only the formula for  $\Delta_{1,1}(E_0^{(p)})$  by induction on  $p$ , as the other formula can be similarly verified. The case for  $p = 1$  follows directly from Lemma 4.3. Assume now the formula for  $\Delta_{1,1}(E_0^{(p)})$  is valid for some  $p$ . Then,

$$\begin{aligned}
& \Delta_{1,1}(E_0^{(p)} E_0) \\
&= \left( \sum_{a+b=p} \pi^b q^{ab} E_1^{(a)} K_1^b \otimes E_1^{(b)} \right) \cdot (E_1 \otimes e_1 + \pi K_1 \otimes E_1) \\
&= \sum_{a+b=p} q^{(a+2)b} [a+1] E_1^{(a+1)} K_1^b \otimes E_1^{(b)} + \sum_{a+b=p} \pi^{b+1} q^{ab} [b+1] E_1^{(a)} K_1^{b+1} \otimes E_1^{(b+1)} \\
&\stackrel{(\star)}{=} [p+1] E_1^{(p+1)} \otimes e_1 + \pi^{p+1} [p+1] K_1^{p+1} \otimes E_1^{(p+1)} \\
&\quad + \sum_{a+b=p, a \geq 1, b \geq 1} (q^{(a+1)(b+1)} [a] + \pi^{b+1} q^{ab} [b+1]) E_1^{(a)} K_1^{b+1} \otimes E_1^{(b+1)} \\
&= [p+1] \sum_{a+b=p+1} \pi^b q^{ab} E_1^{(a)} K_1^b \otimes E_1^{(b)}.
\end{aligned}$$

The identity  $(\star)$  above is obtained by shifting  $a$  to  $a-1$  and  $b$  to  $b+1$  in the first  $\sum$  on the left-hand side. This completes the proof.  $\square$

**4.4. Tensor of Modules.** Let  $M$  and  $N$  be  $U$ -modules. Then  $M \otimes N$  is a  $U \otimes U$ -module via the action

$$(u \otimes v)(m \otimes n) = \pi^{p(v)p(m)}(um) \otimes (vn)$$

for  $\mathbb{Z}_2$ -homogeneous  $v \in U$  and  $m \in M$ . Composition with the coproduct  $\Delta$  defines a  $U$ -module structure on  $M \otimes N$ .

**Example 4.7.** Consider the tensor module  $M = L(1, +) \otimes L(2, +)$ , for which we need only consider the action of  $U_1$  under the coproduct  $\Delta_{1,0}$ . Let  $w$  be a highest weight vector of  $L(1, +)$  and  $v$  be a highest weight vector of  $L(2, +)$ . Then  $M \cong L(3, +) \oplus L(1, +)$ . Indeed, the vector

$$F_1 v \otimes w - \pi q^{-1} [2]^{-1} v \otimes F_0 w$$

is a singular vector generating a copy of  $L(1, +)$  since

$$\Delta_{1,0}(E_1)(F_1 v \otimes w) = E_1 F_1 v \otimes w + \pi^{p(E_1)p(F_1 v)} (\pi K_1) F_1 v \otimes E_1 w = v \otimes w,$$

$$\Delta_{1,0}(E_1)(v \otimes F_0 w) = E_1 v \otimes w + \pi^{p(E_1)p(v)} (\pi K_1) v \otimes E_0 F_0 w = \pi q [2] v \otimes w.$$

## 5. QUASI- $R$ -MATRIX OF $U$

**5.1. Quasi- $R$  matrix.** We can define the quasi- $R$ -matrix  $\Theta$  in our setting (cf. [12, Chapter 4] or [6, Chapter 7] for  $U_q(\mathfrak{sl}(2))$ ). Set

$$a_n = (-1)^n [n]! (\pi q)^{-\binom{n}{2}} (\pi q - q^{-1})^n \in \mathcal{A}, \quad \text{for } n \geq 0. \quad (5.1)$$

(Compare the definition of  $a_n$  with (4.2).) Let  $\epsilon_1, \epsilon_2 \in \{0, 1\}$ . We formally set

$$\Theta_{\epsilon_1, \epsilon_2} = \sum_{n \geq 0} \Theta_{\epsilon_1, \epsilon_2}^n, \quad \text{with } \Theta_{\epsilon_1, \epsilon_2}^n = a_n F_{\epsilon_1}^{(n)} \otimes E_{\epsilon_2}^{(n)},$$

where  $E_\epsilon^{(0)} = F_\epsilon^{(0)} = e_\epsilon$ , the idempotent corresponding to  $U_\epsilon$ . Then  $\Theta_{\epsilon_1, \epsilon_2}$  lies in some completion of  $U_{\epsilon_1} \otimes U_{\epsilon_2}$ , and it can be regarded as a well-defined linear operator on the tensor product of finite-dimensional weight  $U$ -modules. Below we denote  $u_1 \otimes u_2 = \overline{u_1} \otimes \overline{u_2}$  for  $u_1, u_2 \in U$  and set  $\overline{\Delta} = \overline{\phantom{x}} \circ \Delta \circ \overline{\phantom{x}}$ .

**Proposition 5.1.** *Let  $\epsilon_1, \epsilon_2 \in \{0, 1\}$ , and let  $u \in U_{\epsilon_1 + \epsilon_2}$ . Then*

- (1)  $\Delta_{\epsilon_1, \epsilon_2}(u)\Theta_{\epsilon_1, \epsilon_2} = \Theta_{\epsilon_1, \epsilon_2}\overline{\Delta}_{\epsilon_1, \epsilon_2}(u);$
- (2)  $\Theta_{\epsilon_1, \epsilon_2}\overline{\Theta}_{\epsilon_1, \epsilon_2} = e_{\epsilon_1} \otimes e_{\epsilon_2}.$

*Proof.* To avoid cumbersome notation, we will drop the subscripts on  $E, F, K$ ; the hidden subscripts can be recovered from the positions in the tensors.

(1) If  $\Delta_{\epsilon_1, \epsilon_2}(u_1)\Theta_{\epsilon_1, \epsilon_2} = \Theta_{\epsilon_1, \epsilon_2}\overline{\Delta}_{\epsilon_1, \epsilon_2}(u_1)$  and  $\Delta_{\epsilon_1, \epsilon_2}(u_2)\Theta_{\epsilon_1, \epsilon_2} = \Theta_{\epsilon_1, \epsilon_2}\overline{\Delta}_{\epsilon_1, \epsilon_2}(u_2)$ , then clearly  $\Delta_{\epsilon_1, \epsilon_2}(u_1 u_2)\Theta_{\epsilon_1, \epsilon_2} = \Theta_{\epsilon_1, \epsilon_2}\overline{\Delta}_{\epsilon_1, \epsilon_2}(u_1 u_2)$ . Hence it suffices to check (1) on the generators  $E, F, K$ , which is equivalent to proving the following identities:

- (i)  $(E \otimes e)\Theta_{\epsilon_1, \epsilon_2}^n + (\pi^{\epsilon_1} K \otimes E)\Theta_{\epsilon_1, \epsilon_2}^{n-1} = \Theta_{\epsilon_1, \epsilon_2}^n(E \otimes e) + \Theta_{\epsilon_1, \epsilon_2}^{n-1}(K^{-1} \otimes E);$
- (ii)  $(e \otimes F)\Theta_{\epsilon_1, \epsilon_2}^n + (F \otimes K^{-1})\Theta_{\epsilon_1, \epsilon_2}^{n-1} = \Theta_{\epsilon_1, \epsilon_2}^n(e \otimes F) + \Theta_{\epsilon_1, \epsilon_2}^{n-1}(F \otimes \pi^{\epsilon_2} K);$
- (iii)  $(K \otimes K)\Theta_{\epsilon_1, \epsilon_2}^n = \Theta_{\epsilon_1, \epsilon_2}^n(K \otimes K).$

For (i), we have

$$\begin{aligned} (E \otimes e)\Theta_{\epsilon_1, \epsilon_2}^n - \Theta_{\epsilon_1, \epsilon_2}^n(E \otimes e) &= a_n(EF^{(n)} - \pi^n F^{(n)}E) \otimes E^{(n)} \\ &= \pi^{1-n} F^{(n-1)} a_n \left( \frac{(\pi q)^{1-n} \pi^{\epsilon_1} K - q^{n-1} K^{-1}}{\pi q - q^{-1}} \right) \otimes E^{(n)} \\ &= \frac{\pi^{1-n} a_n}{a_{n-1}[n]} \Theta_{\epsilon_1, \epsilon_2}^{n-1} \left( \frac{(\pi q)^{1-n} \pi^{\epsilon_1} K - q^{n-1} K^{-1}}{\pi q - q^{-1}} \right) \otimes E, \end{aligned}$$

and

$$\begin{aligned} (\pi^{\epsilon} K \otimes E)\Theta_{\epsilon_1, \epsilon_2}^{n-1} - \Theta_{\epsilon_1, \epsilon_2}^{n-1}(K^{-1} \otimes E) \\ = q^{1-n}(\pi q - q^{-1})\Theta_{\epsilon_1, \epsilon_2}^{n-1} \left( \frac{(\pi q)^{1-n} \pi^{\epsilon} K - q^{n-1} K^{-1}}{\pi q - q^{-1}} \right) \otimes E. \end{aligned}$$

Hence (i) follows by applying (5.1).

For (ii), we have

$$\begin{aligned} (e \otimes F)\Theta_{\epsilon_1, \epsilon_2}^n - \Theta_{\epsilon_1, \epsilon_2}^n(e \otimes F) &= a_n F^{(n)} \otimes (\pi^n FE^{(n)} - E^{(n)}F) \\ &= a_n F^{(n)} \otimes \left( \pi^{1-n} E^{(n-1)} \frac{q^{1-n} K^{-1} - (\pi q)^{n-1} \pi^{\epsilon_2} K}{\pi q - q^{-1}} \right) \\ &= \frac{a_n}{a_{n-1}[n]} \Theta_{\epsilon_1, \epsilon_2}^{n-1} F \otimes \left( \frac{q^{1-n} K^{-1} - (\pi q)^{n-1} \pi^{\epsilon_2} K}{\pi q - q^{-1}} \right), \end{aligned}$$

and

$$\begin{aligned} (F \otimes K^{-1})\Theta_{\epsilon_1, \epsilon_2}^{n-1} - \Theta_{\epsilon_1, \epsilon_2}^{n-1}(F \otimes \pi^{\epsilon_2} K) \\ = \pi^{1-n} q^{1-n} (\pi q - q^{-1}) \Theta_{\epsilon_1, \epsilon_2}^{n-1} F \otimes \left( \frac{q^{1-n} K^{-1} - (\pi q)^{n-1} \pi^{\epsilon_2} K}{\pi q - q^{-1}} \right). \end{aligned}$$

Hence (ii) follows. The identity (iii) is clear.

(2) Write the formal product

$$\Theta_{\epsilon_1, \epsilon_2} \overline{\Theta}_{\epsilon_1, \epsilon_2} = \sum_{n \geq 0} b_n F^{(n)} \otimes E^{(n)}.$$

Comparing coefficients, we compute that  $b_0 = 1$ , and for  $n \geq 1$ ,

$$b_n = [n]!(\pi q - q^{-1})^n \sum_{t=0}^n (-1)^t \pi^{n(n-t)} (q^{-1})^{-\binom{t}{2}} (\pi q)^{-\binom{n-t}{2}} \begin{bmatrix} n \\ t \end{bmatrix} = 0,$$

where the last equality follows from a version of  $q$ -binomial identity for super binomial coefficients. Hence  $\Theta_{\epsilon_1, \epsilon_2} \overline{\Theta}_{\epsilon_1, \epsilon_2} = e_{\epsilon_1} \otimes e_{\epsilon_2}$ .  $\square$

Set  $\Theta = \Theta_{0,0} + \Theta_{0,1} + \Theta_{1,0} + \Theta_{1,1}$ .

**Corollary 5.2.** *We have  $\Delta(u)\Theta = \Theta\overline{\Delta}(u)$ , for  $u \in U$ , and  $\Theta\overline{\Theta} = 1 \otimes 1$ .*

Define an antilinear operator

$$\Psi = \Theta \circ (\bar{\phantom{x}} \times \bar{\phantom{x}})$$

on  $M_1 \otimes M_2$  as in [12, 24.3.2], where  $M_1$  and  $M_2$  are finite-dimensional weight  $U$ -modules. The following can be proved as in *loc. cit.*

**Proposition 5.3.** *The operator  $\Psi$  acts as an antilinear involution on the  $\mathbb{Q}(q)$ -vector space  $M_1 \otimes M_2$ , where  $M_1$  and  $M_2$  are finite-dimensional  $U$ -modules.*

**5.2. Canonical basis for  ${}^\omega L(s) \otimes L(t)$ .** Suppose  $M$  is a  $U$ -module. We define  ${}^\omega M$  to be the same vector space as  $M$  but with the  $U$ -module action given by  $u \cdot m = \omega(u)m$ . In particular, a highest weight module becomes a lowest weight module under this transformation. Given  $n \in \mathbb{Z}$ , we define

$$p(n) \in \{0, 1\} \text{ such that } p(n) \equiv n \pmod{2}.$$

Consider the  $U$ -module

$$L(s, t) = {}^\omega L(s) \otimes L(t), \quad \text{for } s, t \in \mathbb{N}.$$

This module has a basis

$$E_{p(s)}^{(a)} \eta \otimes F_{p(t)}^{(b)} \nu, \quad 0 \leq a \leq s, 0 \leq b \leq t,$$

where  $\eta, \nu$  are the lowest weight and highest weight vectors respectively. This basis also generates a  $\mathcal{A}$ -submodule  ${}_{\mathcal{A}}\mathcal{L}(s, t)$  which is also an  ${}_{\mathcal{A}}U$ -module. Note that  $\Theta$  and  $\Psi$  are well defined on  $L(s, t)$  and  ${}_{\mathcal{A}}\mathcal{L}(s, t)$ .

Now we have  $\Psi(E_{p(s)}^{(a)} \eta \otimes F_{p(t)}^{(b)} \nu) = E_{p(s)}^{(a)} \eta \otimes F_{p(t)}^{(b)} \nu + (*)$ , where  $(*)$  is an  $\mathcal{A}$ -linear combination of  $E_{p(s)}^{(i)} \eta \otimes F_{p(t)}^{(j)} \nu$ , with  $(i, j) \prec (a, b)$ . Here the partial order  $\preceq$  on  $\mathbb{N}^2$  is defined by declaring that  $(i, j) \preceq (m, n)$  if and only if  $m - n = i - j$  and  $m \leq i$  (hence also  $n \leq j$ ). Then by a variant of [12, Lemma 24.2.1] adapted to our bar map (2.6), we have the following.

**Proposition 5.4.** *Retain the notations above. There exists a unique  $\Psi$ -invariant element  $(E^{(a)} \diamond F^{(b)})_{s,t} \in {}_{\mathcal{A}}\mathcal{L}(s, t)$ , for  $0 \leq a \leq s, 0 \leq b \leq t$ , such that*

$$(E^{(a)} \diamond F^{(b)})_{s,t} = \sum_{m,n} c_{a,b;m,n}^{s,t} E_{p(s)}^{(m)} \eta \otimes F_{p(t)}^{(n)} \nu,$$

where  $c_{a,b;a,b}^{s,t} = 1$ ,  $c_{a,b;m,n}^{s,t} \in q^{-1}\mathbb{Z}[q^{-1}]$ , for all  $(m, n) \prec (a, b)$ .

This is an analogue of [12, Theorem 24.3.3]. The elements  $(E^{(a)} \diamond F^{(b)})_{s,t}$ , for  $0 \leq a \leq s, 0 \leq b \leq t$ , will be called the *canonical basis* of  $L(s, t)$ . The coefficients  $c_{a,b;m,n}^{s,t}$  will be determined precisely in Corollary 6.3.

## 6. MODIFIED SUPERALGEBRA AND CANONICAL BASIS

**6.1. Algebra  $\dot{U}$ .** Let  $a, b \in \mathbb{Z}$ , and consider the subspace of  $U$ :

$${}_a J_b = (K_{p(a)} - q^a 1)U_{p(a)} + U_{p(a)}(K_{p(b)} - q^b 1).$$

Then  ${}_a J_b$  is a subspace of  $U_{p(a)}$ , and  ${}_a J_b = U_{p(a)}$  if  $p(a) \neq p(b)$ . We set

$${}_a U_b = U_{p(a)} / {}_a J_b.$$

Note that  ${}_a U_b = \{0\}$  if  $p(a) \neq p(b)$ .

We define

$$\dot{U} = \bigoplus_{m,n \in \mathbb{Z}} {}_m U_n.$$

This is called the *modified* (also called *idempotentized*) quantum enveloping algebra of  $\mathfrak{osp}(1|2)$  (cf. [2, 12]). Let  $p_{m,n} : U \rightarrow {}_m U_n$  be the canonical projection. We endow  $\dot{U}$  with the structure of an associative algebra under the multiplication

$$p_{k,\ell}(x)p_{m,n}(y) = \delta_{\ell,m}p_{k,n}(xy), \quad \text{for } x, y \in U; \ k, \ell, m, n \in \mathbb{Z}. \quad (6.1)$$

The algebra  $\dot{U}$  inherits a  $\mathbb{Z}$ -grading from  $U$ :

$$\dot{U} = \bigoplus_{k \in 2\mathbb{Z}} \dot{U}(k),$$

where

$$\dot{U}(k) = \sum_{m,n \in \mathbb{Z}} p_{m,n}(U(k)).$$

Note that if  $x \in U(2i)$ , then  $p_{m,n}(x) = 0$  if  $2i \neq m - n$ , since the identity  $q^{2i}x = KxK^{-1}$  in  $U$  descends to  $q^{2i}p_{m,n}(x) = q^{m-n}p_{m,n}(x)$ . The new feature in this algebra is the addition of idempotents  $1_n = p_{n,n}(1)$ , which satisfy

$$1_m 1_n = \delta_{m,n} 1_n.$$

We have

$${}_m U_n = 1_m \dot{U} 1_n.$$

Also, we have that  $\dot{U} = \dot{U}_0 \oplus \dot{U}_1$ , where

$$\dot{U}_\epsilon = \sum_{a,b \in \mathbb{Z}} 1_{2a+\epsilon} \dot{U} 1_{2b+\epsilon}.$$

Moreover,  $\dot{U}_0$  and  $\dot{U}_1$  are subalgebras of  $\dot{U}$  such that  $\dot{U}_0 \dot{U}_1 = \dot{U}_1 \dot{U}_0 = 0$ .

**6.2.  $\dot{U}$  as a  $U$ -bimodule.** The algebra  $\dot{U}$  has a natural  $U$ -bimodule structure: if  $x \in U(k)$ ,  $y \in \dot{U}$  and  $z \in U(n)$  then we set

$$xp_{\ell,m}(y)z = p_{k+\ell,m-n}(xyz). \quad (6.2)$$

With this action, we have the following identities in  $\dot{U}$ , for  $n \in \mathbb{Z}$ ,  $a \in \mathbb{N}$ ,  $\epsilon = 0, 1$ :

$$\begin{aligned} E_\epsilon^{(a)} 1_n &= \delta_{\epsilon,p(n)} 1_{n+2a} E_\epsilon^{(a)}, & F_\epsilon^{(a)} 1_n &= \delta_{\epsilon,p(n)} 1_{n-2a} F_\epsilon^{(a)}, \\ (E_\epsilon F_\epsilon - \pi F_\epsilon E_\epsilon) 1_n &= \delta_{\epsilon,p(n)} [n] 1_n, \end{aligned} \quad (6.3)$$

$$[K_\epsilon; m] 1_n = \delta_{\epsilon,p(n)} [n+m] 1_n, \quad \begin{bmatrix} K_\epsilon; m \\ a \end{bmatrix} 1_n = \delta_{\epsilon,p(n)} \begin{bmatrix} m+n \\ a \end{bmatrix} 1_n. \quad (6.4)$$

The following is a super analogue of [12, 23.1.3].

**Proposition 6.1.** *The following identities hold in  $\dot{U}$ : for  $n \in \mathbb{Z}$ ,  $r, s \geq 0$ ,*

$$\pi^{rs} E_\epsilon^{(r)} 1_n F_\epsilon^{(s)} = \delta_{\epsilon,p(n)} \sum_{i=0}^{\min(r,s)} \pi^{\binom{i+1}{2}} \begin{bmatrix} n+(r+s) \\ i \end{bmatrix} F_\epsilon^{(s-i)} 1_{n+2s+2r-2i} E_\epsilon^{(r-i)}, \quad (6.5)$$

$$\pi^{rs} F_\epsilon^{(s)} 1_n E_\epsilon^{(r)} = \delta_{\epsilon,p(n)} \sum_{i=0}^{\min(r,s)} \pi^{\binom{i}{2}+\epsilon i} \begin{bmatrix} (r+s)-n \\ i \end{bmatrix} E_\epsilon^{(r-i)} 1_{n-2s-2r+2i} F_\epsilon^{(s-i)}. \quad (6.6)$$

*Proof.* First, it is clear by definition that the expressions are zero unless the parities match, so we may assume that  $\epsilon = p(n)$ . Using (6.3), (6.4) and Lemma 2.8, we compute that

$$\begin{aligned} \pi^{rs} E_\epsilon^{(r)} 1_n F_\epsilon^{(s)} &= \left( \sum_{i=0}^{\min(r,s)} \pi \binom{i+1}{2} F_\epsilon^{(s-i)} \begin{bmatrix} K_\epsilon; 2i - (r+s) \\ i \end{bmatrix} E_\epsilon^{(r-i)} \right) 1_{n+2s} \\ &= \sum_{i=0}^{\min(r,s)} \pi \binom{i+1}{2} F_\epsilon^{(s-i)} \begin{bmatrix} K_\epsilon; 2i - (r+s) \\ i \end{bmatrix} 1_{n+2s+2r-2i} E_\epsilon^{(r-i)} \\ &= \sum_{i=0}^{\min(r,s)} \pi \binom{i+1}{2} \begin{bmatrix} n + (r+s) \\ i \end{bmatrix} F_\epsilon^{(s-i)} 1_{n+2s+2r-2i} E_\epsilon^{(r-i)}. \end{aligned}$$

This proves (6.5). The identity (6.6) can be proved similarly, using in addition the identities (2.3).  $\square$

**6.3. Additional structures of  $\dot{U}$ .** We also note that  $\dot{U}$  has a triangular decomposition as in Lusztig [12, 23.2]. Recall the algebra  $\mathbf{f}$  from §4.2. The  $U$ -bimodule structure induces a  $(\mathbf{f}, \mathbf{f}^{\text{op}})$ -bimodule structure on  $\dot{U}$  via

$$x \otimes y \cdot u = x^- u y^+, \quad \text{for } x, y \in \mathbf{f}, u \in \dot{U}.$$

Recall that  $F_\epsilon^{(a)} 1_n E_\epsilon^{(b)} = 0 = E_\epsilon^{(b)} 1_n F_\epsilon^{(a)}$  if and only if  $\epsilon \neq p(n)$ . Hence we adopt the following convention by dropping the subscript  $\epsilon$  without ambiguity:

$$F^{(a)} 1_n E^{(b)} := F_{p(n)}^{(a)} 1_n E_{p(n)}^{(b)}, \quad E^{(a)} 1_n F^{(b)} := E_{p(n)}^{(a)} 1_n F_{p(n)}^{(b)}. \quad (6.7)$$

In this way, we could also drop all subscripts  $\epsilon$  as well as  $\delta_{\epsilon, p(n)}$  in (6.3)-(6.6).

It follows by the triangular decomposition of  $U$  that the elements  $F^{(a)} 1_n E^{(b)}$ , for  $n \in \mathbb{Z}, a, b \in \mathbb{N}$ , form a basis for  $\dot{U}$ . Similarly,  $E^{(b)} 1_n F^{(a)}$ , for  $n \in \mathbb{Z}, a, b \in \mathbb{N}$  form a basis for  $\dot{U}$ . In addition, it is clear from (6.5) and (6.6) that these two bases span the same  $\mathcal{A}$ -submodule of  $\dot{U}$ , denoted by  ${}_{\mathcal{A}}\dot{U}$ . This  $\mathcal{A}$ -submodule  ${}_{\mathcal{A}}\dot{U}$  is in fact an  $\mathcal{A}$ -subalgebra generated by the elements  $E^{(a)} 1_n$  and  $F^{(a)} 1_n$ , for  $n \in \mathbb{Z}, a \in \mathbb{N}$ .

We say a  $\dot{U}$ -module is *unital* if for every  $v \in M$ ,  $1_n v = 0$  for all but finitely many  $n \in \mathbb{Z}$  and  $v = \sum_{n \in \mathbb{Z}} 1_n v$ . Each unital module is a weight  $U$ -module under the action

$u \cdot v = \sum_{n \in \mathbb{Z}} (u 1_n) v$ , where  $u 1_n$  is viewed as an element of  $\dot{U}$ . Likewise, each weight  $U$ -

module with weights in  $q^{\mathbb{Z}}$  is naturally a unital  $\dot{U}$ -module: given a weight decomposition  $v = \sum_{n \in \mathbb{Z}} v_n$  such that  $K v_n = q^n v_n$ , we set  $1_n v = v_n$ .

We define  $\Delta_{a,b,c,d} : {}_{a+b}U_{c+d} \rightarrow {}_aU_c \otimes {}_bU_d$  by (cf. [12, 23.1.5])

$$\Delta_{a,b,c,d}(p_{a+b,c+d}(x)) = (p_{a,c} \otimes p_{b,d}) \circ \Delta(x).$$

The direct product of these maps for various  $a, b, c, d$  defines a coproduct on  $\dot{U}$  which restricts to  $\mathcal{A}$ -linear homomorphism on  ${}_{\mathcal{A}}\dot{U}$ .

The antilinear bar-involution  $\bar{\phantom{x}} : U \rightarrow U$  induces an antilinear bar-involution  $\bar{\phantom{x}} : \dot{U} \rightarrow \dot{U}$ , which fixes each idempotent  $1_n$  for  $n \in \mathbb{Z}$ , and satisfies  $\overline{xy} = \bar{y}\bar{x}$  for  $x, y \in U$  and  $h \in \dot{U}$ . Similarly, the (anti-)automorphisms  $\omega, \tau$  and  $\rho$  on  $U$  induce (anti-)automorphisms on  $\dot{U}$  (denoted by the same letters), which respect the  $U$ -bimodule structure, and  $\rho(1_n) = 1_n$ ,  $\omega(1_n) = 1_{-n}$ ,  $\tau(1_n) = 1_{-n}$ , for  $n \in \mathbb{Z}$ .



**6.4. Canonical basis for  $\dot{U}$ .** Following Lusztig [12], a canonical basis for  $\dot{U}$  should be a bar-invariant  $\mathbb{Q}(q)$ -basis for  $\dot{U}$  and an  $\mathcal{A}$ -basis for  ${}_{\mathcal{A}}\dot{U}$  which consist of elements of the form

$$u = E^{(a)} \diamond_k F^{(b)} \in {}_{\mathcal{A}}\dot{U}1_k, \text{ for } a, b \in \mathbb{N}, k \in \mathbb{Z},$$

such that  $u(\eta \otimes \nu) = (E^{(a)} \diamond F^{(b)})_{s,t}$  where  $\eta$  is the lowest weight vector for  ${}^{\omega}L(s)$  and  $\nu$  is the highest weight vector for  $L(t)$ , with  $t - s = k$ . We take this as the definition of a canonical basis for  $\dot{U}$ .

Keeping in mind the convention (6.7), we consider the elements

$$E^{(a)}1_{-n}F^{(b)}, \quad \pi^{ab}F^{(b)}1_nE^{(a)}, \quad \text{for } a, b \in \mathbb{N}, n \in \mathbb{Z}, n \geq a + b. \quad (6.8)$$

By (6.5), we have the following overlapping elements in (6.8):

$$E^{(a)}1_{-n}F^{(b)} = \pi^{ab}F^{(b)}1_nE^{(a)}, \quad \text{for } n = a + b. \quad (6.9)$$

The following is a super analogue of [12, Proposition 25.3.2], and it formally looks identical!

**Theorem 6.2.** *The elements in (6.8) subject to the identification (6.9) form a canonical basis for  $\dot{U}$ . Moreover, if  $n \geq a + b$ , we have*

$$\begin{aligned} E^{(a)}1_{-n}F^{(b)} &= E^{(a)} \diamond_{2b-n} F^{(b)}, \\ \pi^{ab}F^{(b)}1_nE^{(a)} &= E^{(a)} \diamond_{n-2a} F^{(b)}. \end{aligned}$$

*Proof.* First, recall that all elements of the form  $E^{(a)}1_nF^{(b)}$  form a basis for the  $\mathcal{A}$ -algebra  ${}_{\mathcal{A}}\dot{U}$  and  $\mathbb{Q}(q)$ -algebra  $\dot{U}$ . If  $a + b > n$ ,  $E^{(a)}1_{-n}F^{(b)}$  can be expressed as a  $\mathcal{A}$ -linear combination of the elements in (6.8) by using (6.5) as follows:

$$\pi^{ab}E^{(a)}1_{-n}F^{(b)} = \sum_{i=0}^{\min(a,b)} \pi^{\binom{i+1}{2}} \begin{bmatrix} a+b-n \\ i \end{bmatrix} F^{(b-i)}1_{2a+2b-n-2i}E^{(a-i)},$$

where  $0 \leq a - i + b - i < (a + b - n) + a + b - 2i = 2a + 2b - n - 2i$ . Hence we conclude that the set (6.8) forms a spanning set of  $\dot{U}$ . On the other hand, the set (6.8) naturally splits into two halves, each of which is already linearly independent. Except for the case  $a + b = n$  with identification (6.9), the halves live in different subspaces  ${}_aU_b$  and hence are necessarily linearly independent. This shows the linear independence of the set (6.8) subject to the identification (6.9).

Let  $\eta_s$  and  $\nu_t$  be the lowest and highest weight vectors of  ${}^{\omega}L(s)$  and  $L(t)$ . We have  $E^{(a)}1_{-n}F^{(b)}(\eta_s \otimes \nu_t) = 0$  unless  $-n + 2b = t - s$ , in which case we compute by Lemma 4.6

that

$$\begin{aligned}
& E^{(a)} 1_{-n} F^{(b)} (\eta_s \otimes \nu_t) \\
&= \Delta(E^{(a)}) \Delta(F^{(b)}) (\eta_s \otimes \nu_t) = \Delta(E^{(a)}) (\eta_s \otimes F^{(b)} \nu_t) \\
&= \sum_{a=c+d} \pi^{sd} q^{cd} E^{(c)} K^d \eta_s \otimes E^{(d)} F^{(b)} \nu_t \\
&= \sum_{a=c+d} \pi^{sd} q^{dc-ds} E^{(c)} \eta_s \otimes E^{(d)} F^{(b)} \nu_t \\
&= \sum_{a=c+d} \sum_{i=0}^{\min(b,d)} \pi^{sd} q^{dc-ds} E^{(c)} \eta_s \otimes \pi^{-bd} \pi^{\binom{i+1}{2}} F^{(b-i)} \left[ \begin{matrix} K; 2i - (b+d) \\ i \end{matrix} \right] E^{(d-i)} \nu_t \\
&= \sum_{a=c+d} \pi^{sd} q^{dc-ds} E^{(c)} \eta_s \otimes \pi^{-bd} \pi^{\binom{d+1}{2}} \left[ \begin{matrix} d-b+t \\ d \end{matrix} \right] F^{(b-d)} \nu_t \\
&= \sum_{0 \leq j \leq \min(a,b)} \pi^{sj + \binom{j+1}{2} - bj} q^{j(a-j-s)} \left[ \begin{matrix} j-b+t \\ j \end{matrix} \right] E^{(a-j)} \eta_s \otimes F^{(b-j)} \nu_t.
\end{aligned}$$

Let us denote by  $X$  the right-hand side of the last equation. Then  $X$  is bar-invariant since the left-hand side is; it is also therefore  $\Theta$ -invariant since  $\Theta(\eta_s \otimes \nu_t) = \eta_s \otimes \nu_t$ , so  $X$  is  $\Psi$ -invariant. The leading term (i.e., the term with  $j = 0$ ) of  $X$  is  $E^{(a)} \eta_s \otimes F^{(b)} \nu_t$ . If  $j > 0$ , a degree argument shows that  $q^{j(a-j-s)} \left[ \begin{matrix} j-b+t \\ j \end{matrix} \right]$  lies in  $q^{-1} \mathbb{Z}[q^{-1}]$ . Hence  $X$  satisfies the defining properties of the element  $(E^{(a)} \diamond F^{(b)})_{s,t}$  (see Proposition 5.4), and then must be equal. A similar argument applies to  $F^{(b)} 1_n E^{(a)}$ .

It is clear from the triangular decomposition and the definition of  $\bar{\cdot}$  that the other properties of a canonical basis are satisfied, completing the proof.  $\square$

From the proof above, we have the following formula for the coefficients  $c_{a,b;m,n}^{s,t}$  in the expansion of  $(E^{(a)} \diamond F^{(b)})_{s,t}$  as defined in Proposition 5.4.

**Corollary 6.3.** *Let  $0 \leq a \leq s$ ,  $0 \leq b \leq t$ . For  $0 \leq j \leq \min(a, b)$ , we have*

$$c_{a,b;a-j,b-j}^{s,t} = \pi^{sj + \binom{j+1}{2} - bj} q^{j(a-j-s)} \left[ \begin{matrix} j-b+t \\ j \end{matrix} \right].$$

**6.5. A bilinear form on  $\dot{U}$ .** Recall the definition of  $\rho$  from Proposition 2.7. Since we have defined a suitable bilinear form  $(\cdot, \cdot)$  on  $\mathbf{f}$  (see (4.1) and (4.2)) and constructed the canonical basis on  $\dot{U}$ , the same proof in [12, 26.1.2] leads to the following.

**Proposition 6.4.** *There exists a unique bilinear form  $(\cdot, \cdot) : \dot{U} \times \dot{U} \rightarrow \mathbb{Q}(q)$  such that*

- (1)  $(1_a x 1_b, 1_c y 1_d) = 0$  whenever  $a \neq c$  or  $b \neq d$ ,  $a, b, c, d \in \mathbb{Z}$ ;
- (2)  $(ux, y) = (x, \rho(u)y)$  for  $u \in U$  and  $x, y \in \dot{U}$ ;
- (3)  $(x^{-1} 1_a, y^{-1} 1_a) = (x, y)$  for all  $x, y \in \mathbf{f}$  and  $a \in \mathbb{Z}$ .

Moreover, the bilinear form  $(\cdot, \cdot)$  is symmetric.

## 7. THE COVERING ALGEBRAS

Essentially all the constructions and results in the previous sections make sense in the framework of covering algebras introduced below by treating  $\pi$  as a formal parameter satisfying  $\pi^2 = 1$ . The idea of (half) covering algebras first appeared in [5]. Given a ring  $A$  with unit, we define a new ring  $A^\pi = A[\pi]/(\pi^2 - 1)$ . We shall mainly need  $\mathcal{A}^\pi$

and  $\mathbb{Q}(q)^\pi$  below. Note that  $\mathcal{A}^\pi \subset \mathbb{Q}(q)^\pi$ . The quantum integers and quantum binomials  $[n]$ ,  $\begin{bmatrix} n \\ i \end{bmatrix}$  in (2.1) and (2.2) make sense as elements in  $\mathcal{A}^\pi$  and also in  $\mathbb{Q}(q)^\pi$ .

**7.1. Covering algebra  $U^\pi$ .** We define the *covering algebra*  $U^\pi$  for  $\mathfrak{osp}(1|2)$  to be the  $\mathbb{Q}(q)^\pi$ -(super)algebra generated by elements  $E_\epsilon, F_\epsilon, K_\epsilon$  and  $K_\epsilon^{-1}$  for  $\epsilon \in \{0, 1\}$ , subject to the relations (1)-(3) in Remark 2.4. Then all the definitions and calculations earlier on can be translated to the covering algebra. Indeed, all computations only involve quotients of elements of the form  $(\pi q)^n - q^{-n}$  and we never used  $1 + \pi = 0$  to reduce any expression. Therefore we have the following.

- (1)  $U^\pi$  is a free  $\mathbb{Q}(q)^\pi$ -module with basis  $F_\epsilon^{(a)} K_\epsilon^b E_\epsilon^{(c)}$  for  $a, c \in \mathbb{N}$ ,  $b \in \mathbb{Z}$ ,  $\epsilon \in \{0, 1\}$ .
- (2)  $U^\pi$  has algebra (anti-)automorphisms as described in Proposition 2.7 which fix  $\pi$ .
- (3) The elements  $E^{(r)}, F^{(s)}$  satisfy the commutation relations in Lemma 2.8.
- (4)  $U^\pi$  has a Hopf superalgebra structure.
- (5)  $U^\pi$  admits a quasi-R matrix  $\Theta$  and the map  $\Psi$  as operators on tensor products of modules.
- (6) Proposition 5.4 remains valid, with  $c_{a,b;m,n}^{s,t} \in q^{-1}\mathbb{N}[q^{-1}, \pi]$ .

**7.2. Covering algebra  $\dot{U}^\pi$ .** Similarly, we can modify the definition of  $\dot{U}$  in §6.1 as follows. Let  $a, b \in \mathbb{Z}$  and set

$${}_a U_b^\pi = U^\pi / \left( (K_{p(a)} - q^a)U^\pi + U^\pi(K_{p(b)} - q^b) \right),$$

and define

$$\dot{U}^\pi = \bigoplus_{a,b \in \mathbb{Z}} {}_a U_b^\pi.$$

This is called the *modified* (also called *idempotentized*) covering quantum (super)algebra of  $U^\pi$ . Imitating the  $\mathcal{A}$ -subalgebra  ${}_A \dot{U}$ , we can define the  $\mathcal{A}^\pi$ -subalgebra  ${}_A \dot{U}^\pi$ . We can now reinterpret earlier results on  $\dot{U}$  in the setting of covering algebra as follows:

- (1) The identities (6.3), (6.4), (6.5), and (6.6) are valid in  ${}_A \dot{U}^\pi$ .
- (2) Theorem 6.2 on canonical basis is valid for  ${}_A \dot{U}^\pi$ .

**7.3. Specializations.** The specialization by setting  $\pi$  to be  $\pm 1$  in the constructions and statements for the covering algebras recovers corresponding results for quantum  $\mathfrak{sl}(2)$  and  $\mathfrak{osp}(1|2)$  simultaneously as follows.

- (1) Specializing  $\pi = -1$ , we obtain that  $U^\pi / \langle \pi + 1 \rangle \cong U$  and  $\dot{U}^\pi / \langle \pi + 1 \rangle \cong \dot{U}$ .
- (2) The canonical basis for  $\dot{U}^\pi$  specializes at  $\pi = -1$  to that for  $\dot{U}$ .
- (3) Specializing  $\pi = 1$ , we obtain that  $U^\pi / \langle \pi - 1 \rangle$  is isomorphic to a direct sum of two copies of the quantum group  $U_q(\mathfrak{sl}(2))$ , and  $\dot{U}^\pi / \langle \pi - 1 \rangle$  is isomorphic to the modified algebra  $\dot{U}_q(\mathfrak{sl}(2))$  in [2, 12].
- (4) The canonical basis for  $\dot{U}^\pi$  specializes at  $\pi = 1$  to that for the modified quantum  $\mathfrak{sl}(2)$  given in [12, Proposition 25.3.2].

*Remark 7.1.* The super sign, being inherent in the structure of superalgebras, rules out the hope of positivity of the structure constants for canonical basis of the quantum superalgebra  $\dot{U}$  in the usual sense. Using (6.3), (6.5) and (6.6), we can show that the structure coefficients from multiplying canonical basis elements in  $\dot{U}^\pi$  lie in  $\mathbb{N}[q, q^{-1}, \pi]$ . So passing to covering algebras restores the positivity.

A categorification of  $\dot{U}^\pi$  and its canonical basis, à la Lauda [10] for modified quantum  $\mathfrak{sl}(2)$ , is expected in a generalized framework of spin nilHecke algebras, with  $\pi$  categorified as a parity shift functor as in [5]. Such a categorification would be relevant to odd Khovanov homology and knot invariants (also compare [3]). Forgetting the  $\mathbb{Z}_2$ -grading and the parity shift functor would lead to a (second) categorification of modified quantum  $\mathfrak{sl}(2)$  and its canonical basis; see (4) above.

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